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# *On the Algebraic Proof of a Certain Series.*[*]

BY EMORY MCCLINTOCK.

The following series was published in 1879 (*American Journal of Mathematics*, II, 108):

$$\log x = y + \frac{2a-1}{2}y^2 + \frac{2a-1}{2}\,\frac{2a-2}{3}y^2 + \ldots. \tag{1}$$

Here $y = x^{1-a} - x^{-a}$. In presenting this series at that time I stated that it might be obtained by the aid of Lagrange's theorem, or by the aid of another expansion-theorem, somewhat simpler than that of Lagrange though less comprehensive, which I published (p. 147) in the same paper; and I added that the "temporary lack" of an algebraic demonstration was "certainly to be regretted." I am now able to supply the desired algebraic proof.

It was shown in the paper referred to that, writing $x^h$ for $x$, (1) is equivalent to

$$\log x = h^{-1}\left(y + \frac{2a-1}{2}y^2 + \ldots\right), \tag{2}$$

where $y = x^{h-ha} - x^{-ha}$; that when $a = 0$ and $a = 1$ respectively we have as special cases of (1) the known series

$$\log x = (x-1) \quad - \frac{1}{2}(x-1)^2 \quad + \frac{1}{3}(x-1)^3 \quad - \ldots, \tag{3}$$

$$\log x = (1-x^{-1}) + \frac{1}{2}(1-x^{-1})^2 + \frac{1}{3}(1-x^{-1})^3 + \ldots; \tag{4}$$

that when $a = m/n$, a proper fraction, the coefficients of $y^n$, $y^{2n}$, etc., disappear; that, in particular, when $a = \frac{1}{2}$, the even powers of $y$ disappear, whence, writing $2t$ for $y$, so that $t = \frac{1}{2}\,(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$,

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[*] An abstract of this paper was read before the New York Mathematical Society on March 6, 1891.

$$\log x = 2\left[t - \frac{1}{2}\,\frac{t^3}{3} + \frac{1}{2}\,\frac{3}{4}\,\frac{t^3}{5} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\,\frac{t^5}{7} + \ldots.\right], \tag{5}$$

a series possibly novel for real values of $x$, though known when $x$ is a certain symbol of operation (the usual series for $z = \text{arc}\sin u$ may be obtained from [5] as a special case if $x = e^{2\iota z}$); that (5) is a remarkably convergent logarithmic series; that if, in (2), $h$ tends towards 0, we have

$$\log x = \lim\,(x^{h-ha} - x^{-ha})/h, \tag{6}$$

a general limit-expression of which $\log x = \lim\,(x^h - 1)/h$ is a known case; and that if $a = -h^{-1}$, and $h$ tends towards 0, $\lim y/h = -\lim ay = \lim\,(x^{h+1} - x)/h = x\lim\,(x^h - 1)/h = x\log x$, so that (2) becomes

$$\log x = v - v^2 + 3v^3/2! - 4^2v^4/3! + 5^3v^5/4! - \ldots., \tag{7}$$

where $v = x\log x$.

As regards the validity of these results, the general series (1) will be found convergent when (using mod to express mere arithmetical value)

$$\text{mod}\,(x-1)\,x^{-a} < \text{mod}\,(a-1)^{a-1}a^{-a};$$

and (2) will be found convergent when

$$\text{mod}\,y < \text{mod}\,(a-1)^{a-1}a^{-a}, \quad \text{where} \quad y = x^{h-ha} - x^{-ha}.$$

Applying this statement to special cases, we may say that (1) is convergent if $a \geqq 1$ when $x > 1$, or if $a \leqq 0$ when $0 < x < 1$, unless perhaps when $x - a/(a-1)$, in which case $(x-1)\,x^{-a} = (a-1)^{a-1}a^{-a}$; also, that (1) is convergent if $a = \frac{1}{2}$ when $\text{mod}\,y < 2$, that is, when $x$ lies between the two values of $3 \pm 2\sqrt{2}$. In the case ($a = -\infty$) represented in (7), where (2) is modified by putting $a = -h^{-1}$ and diminishing $h$ towards 0, the application of this criterion shows (7) convergent when $\text{mod}\,v < e^{-1}$, because $v = \lim y/h$, and we must have

$$\text{mod}\,y/h < \text{mod}\,(a-1)^{a-1}a^{-a}/h, \quad \text{where } a = -h^{-1},$$

the limit of the second member being $e^{-1}$. It is easy to prove convergency in any of these special cases, or in any case in which $a$ is an integer; and I have assumed the truth of the criterion for all other cases. For elementary presentation, however, it is enough to say that (1) is convergent when $y$ is sufficiently small, that is, when the value of $x$ has a certain range according to the given value of $a$.

The two known series, (3) and (4), are, one or the other, valid for all values of $x$. The algebraic proof of (1) which I have now to present consists in showing, first, its equivalence to (3) whenever both (1) and (3) are interpretable,* and, secondly, its equivalence to (4) whenever both (1) and (4) are interpretable. If $a$ is assigned, any value of $x$ which makes (1) interpretable will make either (3) or (4) interpretable, so that by this means we have a demonstration of (1) for all interpretable cases.

Let us suppose that (excluding $x = 2$) $x$ and $a$ are such that both (1) and (3) are interpretable. For any such value of $x$ the binomial series for $(1 + z)^{-a}$, where $z = x - 1$, is known to be valid; for, if $z$ is a quantity, mod $z < 1$, and if $z$ is an operation affecting a function of $t$, say $\psi(t)$, it and $\psi(t)$ must be such that $z^{n+1}\psi(t)$ is arithmetically less than $z^n\psi(t)$, the binomial series being interpretable in both cases. Employing that series, therefore, after observing that $y = x^{1-a} - x^{-a} = z(1 + z)^{-a}$, and denoting the second member of (1) by $f(y)$, we have

$$\left.\begin{aligned} f(y) &= y + \frac{2a-1}{2}y^2 + \frac{3a-1}{2}\,\frac{3a-2}{3}y^3 + \ldots . \\ &= z\left(1 - az + \frac{1}{1.2}[a+1]^{2)}z^2 - \ldots .\right) \\ &\quad + \frac{1}{1.2}(2a-1)z^2\left(1 - 2az + \frac{1}{1.2}[2a+1]^{2)}z^2 - \ldots .\right) \\ &\quad + \frac{1}{1.2.3}(3a-1)^{2)}z^3(1 - 3az + \ldots .) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{8}$$

Here $(a + 1)^{2)}$ means $(a + 1)a$, and in general $c^{n)}$ means

$$c(c-1)(c-2)\ldots .(c-n+1).$$

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* A series of quantities is interpretable when it is convergent. A series of symbols of operation is interpretable when, if the operations indicated are performed upon a given function, the result is a convergent series. Thus, the exponential series $1 + x + x^2/2! + \ldots .$ has always a meaning when $x$ is a quantity, but not always when $x$ is the symbol of differentiation (Taylor's theorem). For all series *ad infinitum* (i. e. without remainder) a convention is necessary that the symbols can have no values assigned so as to render such series uninterpretable. Subject to this restriction, any power-series convergent for small values of the variable may be handled freely as long as the variable has no meaning assigned to it.

Collecting the several terms composing the coefficient of $z^m$, we find them to be identical with the several terms of $\phi(m, ma-1)/m! - (-1)^m/m$, where $\phi(m, v)$ means $v^{m-1)} - m(v-k)^{m-1)} + \frac{1}{1.2} m^{2)}(v-2k)^{m-1)} - \ldots$, in which $k = a - 1$. I shall prove that, for all positive integral values of $m$, and for all values of $v$ and $k$, $\phi(m, v) = 0$. Assuming this for the moment, it follows that the coefficient of $z^m$ in (8) is $(-1)^{m-1}/m$, whence

$$f(y) = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \ldots.$$

The second member of this is the second member of the known equation (3); hence, for all values of $y$ which make both (1) and (3) interpretable, $f(y) = \log x$, and (1) is true.

Again, let us suppose that $x$ and $a$ are such that both (1) and (4) are interpretable, in which case the binomial series for $(1-u)^{a-1}$, where $u = 1 - x^{-1}$, is also interpretable. Since $y = x^{1-a} - x^{-a} = u(1-u)^{a-1}$, we shall find, on substituting this for $y$ in $f(y)$, and performing the binomial expansions indicated, that the coefficient of $u^m$ is identical, term by term, with $\phi(m, ma-1) + 1/m$, wherein $k = a$. Since $\phi(m, v) = 0$, the coefficient of $u^m$ is $1/m$, so that

$$f(y) = u + \frac{1}{2} u^2 + \frac{1}{3} u^3 + \ldots.$$

The second member of this is the second member of the known equation (4); hence, for all values of $y$ which make both (1) and (4) interpretable, $f(y) = \log x$, and (1) is again true.

The necessary algebraic proof of the known theorem in finite differences that $\phi(m, v) = 0$ may be supplied as follows. By definition,

$$\phi(m, v) = v^{m-1)} - m(v-k)^{m-1)} + m\frac{m-1}{2}(v-2k)^{m-1)} - \ldots.$$

Let the multiplications indicated in $(v + k - m + 1)\phi(m, v + k)$ and in $(v - mk - m + 1)\phi(m, v)$ be performed, and the first term of the latter series placed beneath the second term of the former, and so on, thus:

$$\begin{aligned}&(v+k)^{m)} - m(v+k-m+1)v^{m-1)} + m\frac{m-1}{2}(v+k-m+1)(v-k)^{m-1)}\\ &\qquad - \ldots.\ (v-mk-m+1)v^{m-1)} - m(v-mk-m+1)(v-k)^{m-1)} + \ldots.\end{aligned}$$

Subtracting the lower series from the upper, term by term, after due consideration of character of the respective progressions, we find the difference to be identical with $\phi(m+1, v+k)$. In brief,

$$\phi(m+1, v+k) = (v+k-m+1)\,\phi(m, v+k) - (v-mk-m+1)\,\phi(m, v). \quad (9)$$

If, then, we know for any given positive integral value of $m$, and for all values of $v$ and $k$, that $\phi(m, v) = 0$, we learn by (9) that the same is true for the value of $m$ next higher, and therefore successively for all higher integral values of $m$. But we may see at once that $\phi(2, v) = 0$, as well as $\phi(1, v) = 0$, for all values of $v$ and $k$, since $\phi(1, v) = 1 - 1$, and $\phi(2, v) = v - 2(v-k) + (v-2k)$; therefore $\phi(m, v) = 0$ for all positive integral values of $m$.